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COVARIANCE AND GAUGE INVARIANCE IN RELATIVISTIC THEORIES OF GRAVITY

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Any metric theory of gravity whose interaction with quantum particles is described by a covariant wave equation is equivalent to a vector theory that satisfies Maxwell-type equations identically. This result does not depend on any particular set of field equations for the metric tensor, but only on covariance. It is derived in the linear case, but can be extended to any order of approximation in the metric deviation. In this formulation of the interaction of gravity with matter, angular momentum and momentum are conserved locally.

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1. Introduction. Solving the Klein-Gordon equation

Utiyama showed ¹ that important general relativistic aspects of the interaction of gravity with matter could be arrived at by generalizing the space-time independent coordinate transformations of special relativity to local ones. These results were later extended by Kibble ² using the complete Poincaré group. More recently Capozziello and De Laurentis ³ have shown that general gauge theories of gravity can be derived from local Poincaré symmetry. Unlike the approach followed in ^{1,2,3}, the present work derives a theory of the interaction of gravity with particles described by covariant wave equations by taking advantage of a restriction to global phase invariance suggested by the covariant wave equations themselves. The solution discussed in this Section does in fact contain gravity in a space-time dependent phase which is the fundamental ingredient of any gauge theory. The case of a particle described by a Klein-Gordon equation is considered in detail, but similar conclusions can be reached starting from other known wave equations.

Covariant wave equations that apply to particles with, or without spin, have solutions ^{4,5,6,7,8} that are exact to first order in the metric deviation $\gamma_{\mu\nu} = g_{\mu\nu} -$

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$\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric, and have been applied to problems like geometrical optics⁷, interferometry and gyroscopy⁵, the spin-flip of particles in gravitational and inertial fields⁹, radiative processes^{10,11} and spin currents¹².

It is useful to re-derive here the solution of the Klein-Gordon equation that, neglecting curvature dependent terms and applying the Lanczos-De Donder condition

$$\gamma_{\alpha\nu}{}^{\nu} - \frac{1}{2}\gamma_{\sigma,\alpha}^{\sigma} = 0, \quad (1)$$

becomes to $\mathcal{O}(\gamma_{\mu\nu})$

$$(\nabla_{\mu}\nabla^{\mu} + m^2)\phi(x) \simeq [\eta_{\mu\nu}\partial^{\mu}\partial^{\nu} + m^2 + \gamma_{\mu\nu}\partial^{\mu}\partial^{\nu}]\phi(x) = 0. \quad (2)$$

The notations and units ($\hbar = c = 1$) are as in⁹. The solution of (2) is obtained by solving the Volterra equation

$$\phi(x) = \phi_0(x) - \int_P^x d^4x' G(x, x') \gamma_{\mu\nu}(x') \partial'^{\mu} \partial'^{\nu} \phi(x'), \quad (3)$$

along the particle world-line, where P is a fixed reference point, x a generic point in the physical future along the world-line, $G(x, x')$ is the causal Green function with $(\partial^2 + m^2)G(x, x') = \delta^4(x - x')$. The free Klein-Gordon equation is

$$(\partial^2 + m^2)\phi_0 = 0. \quad (4)$$

In first approximation ϕ_0 can be substituted for ϕ in (3) and the integrations can then be carried out using the equations

$$(\partial_x^2 + m^2) \frac{1}{2} \int_P^x dz^{\lambda} \gamma_{\alpha\lambda}(z) \partial^{\alpha} \phi_0(x) = \gamma_{\alpha\beta}(x) \partial^{\alpha} \partial^{\beta} \phi_0(x) + \frac{1}{2} \gamma_{\alpha\mu}{}^{\mu}(x) \partial^{\alpha} \phi_0(x), \quad (5)$$

from which $\gamma_{\alpha\beta}(x) \partial^{\alpha} \partial^{\beta} \phi_0(x)$ can be obtained, and

$$\begin{aligned} & (\partial_x^2 + m^2) \frac{1}{2} \int_P^x dz^{\alpha} \int_P^x dz^{\lambda} (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) \partial^{\beta} \phi_0(x) \\ &= (\partial_x^2 + m^2) \frac{1}{2} \int_P^x dz^{\lambda} (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) (x^{\alpha} - z^{\alpha}) \partial^{\beta} \phi_0(x) \\ &= \frac{1}{2} \left(\gamma_{\mu,\beta}^{\mu}(x) - \gamma_{\beta\mu}{}^{\mu}(x) \right) \partial^{\beta} \phi_0(x). \end{aligned} \quad (6)$$

Equations (5) and (6) can be proven by straightforward differentiation. The latter is the four-dimensional extension of a known formula¹³. To $\mathcal{O}(\gamma_{\mu\nu})$ the solution of (2) is

$$\phi(x) = \left(1 - i\hat{\Phi}_G(x)\right) \phi_0(x), \quad (7)$$

where the operator $\hat{\Phi}_G$ is defined as⁹

$$\hat{\Phi}_G(x) = -\frac{1}{2} \int_P^x dz^{\lambda} (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) (x^{\alpha} - z^{\alpha}) \hat{k}^{\beta} \quad (8)$$

$$+\frac{1}{2}\int_P^x dz^\lambda \gamma_{\alpha\lambda} \hat{k}^\alpha,$$

$\hat{k}^\alpha = i\partial^\alpha$ and k^α is the momentum of the plane wave solution ϕ_0 of (4) satisfying $k_\alpha k^\alpha = m^2$. The solution is independent of any field equations for $\gamma_{\mu\nu}$. Equations (7) and (8) are the byproduct of covariance (minimal coupling) and, ultimately, of Lorentz invariance and can therefore be applied to general relativity, in particular to theories in which acceleration has an upper limit^{14,15,16,17,18,19,20,21} and that therefore allow the resolution of astrophysical^{22,23,24,25} and cosmological singularities in quantum theories of gravity^{26,27}. They also are relevant to those theories of asymptotically safe gravity that can be expressed as Einstein gravity coupled to a scalar field²⁸.

The calculation of ϕ can be extended to any order in $\gamma_{\mu\nu}$, but the results cannot be summed up in closed form. In fact, the solution can be written in the form $\phi = \Sigma_h \phi_{(h)}$ where⁵

$$\phi_{(h)}(x) = -i\hat{\Phi}_G \phi_{(h-1)}(x) \quad (9)$$

$$= \frac{1}{2} \int_P^x dz^\lambda [(\gamma_{\alpha\lambda,\beta} - \gamma_{\beta\lambda,\alpha})(x^\alpha - z^\alpha) \partial^\beta - \gamma_{\alpha\lambda} \partial^\alpha] \phi_{(h-1)}(x).$$

Any possible non-linearities present in the wave equation can be treated as perturbations, where applicable²⁹. The results also apply, with appropriate changes, to all known covariant wave equations⁴.

2. Maxwell-type equations

By substituting (7) and (8) in (2) one finds that $\nabla_\mu \phi \rightarrow (\nabla_\mu - i\Phi_{G,\mu})\phi$, where

$$\Phi_{G,\mu}(x) = -\frac{1}{2} \int_P^x dz^\lambda (\gamma_{\mu\lambda,\alpha} - \gamma_{\alpha\lambda,\mu}) k^\alpha + \frac{1}{2} \gamma_{\alpha\mu}(x) k^\alpha. \quad (10)$$

It follows that the particle momentum is $p_\mu = k_\mu - \Phi_{G,\mu}$. By using (10), one can also write

$$\begin{aligned} \nabla^\mu &\rightarrow \nabla^\mu - i\partial_x^\mu \left\{ -\frac{1}{2} \int_P^x dz^\lambda [(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))(x^\alpha - z^\alpha) - \gamma_{\beta\lambda}(z)] k^\beta \right\} \\ &= \nabla^\mu - iK^\mu(z, x) \equiv \mathcal{D}^\mu, \end{aligned} \quad (11)$$

where $\Phi_G(x) = \int_P^x dz^\lambda K_\lambda(z, x)$ and

$$K_\lambda(z, x) = -\frac{1}{2} [(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))(x^\alpha - z^\alpha) - \gamma_{\beta\lambda}(z)] k^\beta. \quad (12)$$

In the form (11) of the covariant derivative, the two-point vector $K_\lambda(z, x)$ plays the role of the vector potential in electromagnetism.

The definition of K_λ given by (12) contains a reference to matter through the momentum k_μ of ϕ_0 . The inclusion of the coupling to matter in K_λ does not however

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create problems in a theory of particle-gravity interactions and offers the benefits of dealing with completely gauge invariant quantities (Section 4).

The equations satisfied by K_λ relative to the base point x^α can now be obtained by differentiating (12) with respect to z . One finds

$$F_{\mu\lambda}(z, x) = K_{\lambda,\mu}(z, x) - K_{\mu,\lambda}(z, x) = R_{\alpha\beta\lambda\mu}(z) (x^\alpha - z^\alpha) k^\beta = R_{\mu\lambda\alpha\beta}(z) J^{\alpha\beta}, \quad (13)$$

where $R_{\alpha\beta\lambda\mu}(z) = \frac{1}{2}(\gamma_{\alpha\lambda,\beta\mu} + \gamma_{\beta\mu,\alpha\lambda} - \gamma_{\alpha\mu,\beta\lambda} - \gamma_{\beta\lambda,\alpha\mu})$ is the linearized Riemann tensor satisfying the identity $R_{\mu\nu\sigma\tau} + R_{\nu\sigma\mu\tau} + R_{\sigma\mu\nu\tau} = 0$ and $J^{\alpha\beta} = \frac{1}{2}[(x^\alpha - z^\alpha)k^\beta - k^\alpha(x^\beta - z^\beta)]$ is the angular momentum about the base point x^α . The physical origin of the base event in the two-point tensors (12) and (13) is closely linked to $J^{\alpha\beta}$. The latter tensor would of course contain also the intrinsic angular momentum when referring to particles with spin. Maxwell-type equations

$$F_{\mu\lambda,\sigma} + F_{\lambda\sigma,\mu} + F_{\sigma\mu,\lambda} = 0 \quad (14)$$

and

$$F^{\mu\lambda}_{,\lambda} \equiv j^\mu = (R^{\mu\lambda}_{\alpha\beta} J^{\alpha\beta})_{,\lambda} = R^{\mu\lambda}_{\alpha\beta,\lambda} (x^\alpha - z^\alpha) k^\beta + R^{\mu\lambda}_{\alpha\beta} k^\beta, \quad (15)$$

can be obtained from (13) using the Bianchi identities $R_{\mu\nu\sigma\tau,\rho} + R_{\mu\nu\tau\rho,\sigma} + R_{\mu\nu\rho\sigma,\tau} = 0$. The current j^μ satisfies the conservation law $j^\mu_{,\mu} = 0$. One finds in particular that the "electric" and "magnetic" components of $F_{\mu\nu}$ are

$$E_i = R_{0i\alpha\beta} J^{\alpha\beta}, \quad H_i = \epsilon_{ijk} R^{kj}_{\alpha\beta} J^{\alpha\beta}. \quad (16)$$

Equations (14) and (15) are identities and do not represent additional constraints on $\gamma_{\mu\nu}$. The recombination of ten $\gamma_{\mu\nu}$ into four K_λ may be regarded as an example of hidden symmetry made manifest by the covariance of the wave equation, hence the interaction of the particle with $\gamma_{\mu\nu}$. Knowledge of $\gamma_{\mu\nu}$ is still needed in order to calculate K_λ and that requires the solution of ten equations. One needs in fact the fifteen components $R_{0i23}, R_{0i13}, R_{0i12}, R_{0i02}, R_{0101}, R_{0103}, R_{0203}$ to determine F_{0i} and the six components $R_{1212}, R_{1213}, R_{1313}, R_{1223}, R_{1323}, R_{2323}$ to calculate F_{ij} . The sum must be decreased by one because $R_{0123} + R_{0231} + R_{0312} = 0$ and the total, twenty, corresponds to the number of independent components of the Riemann tensor. In general, therefore, all $\gamma_{\mu\nu}$ are required in order to determine $F_{\mu\nu}$.

The somewhat inverse approach, whereby gravitons are constructed as photon bound states, is discussed in ³⁰.

3. Conservation laws

Conservation laws are better expressed geometrically in integral form. Several of them are derived in a purely classical context in ³¹ using world function and two-point tensor formalisms. It is shown in particular that for any skew-symmetric tensor $S_{\mu\nu}$ and any closed two-space V_2 in space-time spanned by an open V_3 one has

$$\oint_{V_2} S_{\mu\nu} d\tau^{\mu\nu\sigma} = \int_{V_3} S_{\mu\nu,\sigma} d\tau^{\mu\nu\sigma} = \frac{1}{3} \int_{V_3} (S_{\mu\nu,\sigma} + S_{\nu\sigma,\mu} + S_{\sigma\mu,\nu}) d\tau^{\mu\nu\sigma}. \quad (17)$$

On substituting $F_{\mu\nu}$ for $S_{\mu\nu}$ and taking (14) into account, (17) gives

$$\oint_{V_2} F_{\mu\nu} d\tau^{\mu\nu} = \oint_{V_2} R_{\mu\nu\alpha\beta} J^{\alpha\beta} d\tau^{\mu\nu} = 0, \quad (18)$$

which states that the flux of $J^{\alpha\beta}$ relative to a base point x^α out of a closed surface vanishes. The same result can be obtained by integrating K_λ over a closed V_1 spanned by an open V_2 and letting V_1 shrink to a point. It also follows that, when the wave function is nodal, the flux of K_λ

$$\oint_{V_1} dz^\lambda K_\lambda(z, x) = \int_{V_2} R_{\mu\nu\alpha\beta} J^{\alpha\beta} d\tau^{\mu\nu}, \quad (19)$$

is quantized. Finally, the same procedure applied to j_μ gives

$$\oint_{V_1} j_\mu dz^\mu = \int_{V_2} j_{\mu,\nu} d\tau^{\mu\nu} = \int_{V_2} (R_{\alpha\beta\mu}{}^\lambda J^{\alpha\beta})_{,\lambda\nu} d\tau^{\mu\nu} \quad (20)$$

and, by contracting V_1 to a point and using Bianchi identities,

$$\oint_{V_2} (R_{\alpha\beta\mu}{}^\lambda J^{\alpha\beta})_{,\lambda\nu} d\tau^{\mu\nu} = \oint_{V_2} [(R_{\mu\alpha,\beta\nu} - R_{\mu\beta,\alpha\nu}) J^{\alpha\beta} + 2R_{\mu\beta,\nu} k^\beta] d\tau^{\mu\nu} = 0, \quad (21)$$

that again states that the total flux of angular momentum and momentum out of a closed V_2 vanishes. Since V_2 can be chosen arbitrarily small and j^μ is conserved, one concludes that the total flux of momentum and angular momentum is conserved locally. This is shown in ³ to be a necessary ingredient of gauge theories of gravity. It appears here in a context still independent of any choice of gravitational field equations. Local conservation requires the introduction of the field K_λ . The corresponding force field is represented by (13) that contains curvature in an essential way.

4. Gauge transformations and equations of motion

There are two types of "gauge" transformations at play in what follows. First are those that follow from (13) and leave $F_{\mu\nu}$ invariant under the changes

$$K'_\mu(z, x) = K_\mu(z, x) - \Lambda_{,\mu}(z). \quad (22)$$

It also follows from (11) and (22) that $e^{i\Lambda(z)} \mathcal{D}_\mu e^{-i\Lambda(z)} = \partial_\mu - iK'_\mu(z, x) = \mathcal{D}'_\mu$. These are the usual gauge transformations one must expect in a theory formally analogous to electromagnetism with symmetry group $U(1)$. They can also be used to eliminate redundant terms from the equations, as follows. Differentiating (12) twice with respect to z^μ one finds

$$\partial^2 K_\lambda = -\frac{1}{2} [(\partial^2 \gamma_{\alpha\lambda,\beta} - \partial^2 \gamma_{\beta\lambda,\alpha})(x^\alpha - z^\alpha) + \partial^2 \gamma_{\beta\lambda} - \partial^2 \gamma_{,\lambda}] k^\beta, \quad (23)$$

where $\partial^2 = \partial_\mu \partial^\mu$ and $\gamma = \gamma_\mu{}^\mu$. The last term can be eliminated from (23) by using (22) with $\Lambda = -\frac{1}{2} \gamma_{,\beta} k^\beta$. The corresponding equation for $F_{\lambda\mu}$ can be obtained by

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differentiating (13) and using (1)

$$\begin{aligned} \partial^2 F_{\lambda\mu} = & \frac{1}{4} \left[\partial^2 (-\eta_{\alpha\mu} \gamma_{,\beta\lambda} - \eta_{\beta\lambda} \gamma_{,\alpha\mu} + \eta_{\alpha\lambda} \gamma_{\beta\mu} + \eta_{\beta\mu} \gamma_{,\alpha\lambda}) \right] (x^\alpha - z^\alpha) k^\beta \\ & - \frac{1}{2} \partial^2 (-\eta_{\beta\lambda} \gamma_{,\mu} + \eta_{\beta\mu} \gamma_{,\lambda\beta}) k^\beta, \end{aligned} \quad (24)$$

and is invariant under (22). The second instance in which the term "gauge transformation" is traditionally used is related to the fact that (1) does not determine the coordinates completely. The additional, allowable transformations $z^\alpha \rightarrow z^\alpha + \xi^\alpha$, where ξ^α are first order quantities, induce the changes

$$\gamma_{\alpha\lambda} \rightarrow \gamma_{\alpha\lambda} - \xi_{\alpha,\lambda} - \xi_{\lambda,\alpha}. \quad (25)$$

Because K_λ contains $\gamma_{\mu\nu}$ explicitly, one must ascertain that K_λ and the equations it satisfies behave in a way that is consistent with (25) and (22). One finds that (25) still leaves (23) invariant provided $\partial^2 \xi_\alpha = 0$ and affects K_λ in a way consistent with (22). In fact, by applying (25) to K_λ one obtains $K'_\lambda = K_\lambda - \partial_\lambda \left\{ \frac{1}{2} (-\xi_{\alpha,\beta} + \xi_{\beta,\alpha}) (x^\alpha - z^\alpha) k^\beta \right\}$ as required by (13).

Bel suggested³² an alternative way to strengthen the analogy between electromagnetism and linearized gravity by introducing the tensor $F_{\alpha\beta\lambda} = \gamma_{\beta\lambda,\alpha} - \gamma_{\alpha\lambda,\beta}$ that also satisfies Maxwell-type equations. However $F_{\alpha\beta\lambda}$ transforms under (25) as $F_{\alpha\beta\lambda} \rightarrow F_{\alpha\beta\lambda} + \partial_\lambda (\xi_{\beta,\alpha} - \xi_{\alpha,\beta})$ and is not therefore a gauge invariant quantity. Interesting results can nonetheless be obtained by restricting (25) to the special functions $\xi_\mu = \partial_\mu \xi$ ³². This additional restriction is, of course, unnecessary when dealing with K_λ and $F_{\alpha\beta}$.

Suppose now that $\gamma_{\mu\nu} = \varepsilon_{\mu\nu} \exp(ip_\alpha z^\alpha)$ and that the wave propagates in the direction characterized by the wave vector $p^1 = p^2 = 0$ and $p^3 = p^0 > 0$. Then (1) and (25) provide, among the surviving components, the relations $\varepsilon_{22} =, \varepsilon_{11}, \varepsilon_{02} = -\varepsilon_{32}, \varepsilon_{01} = -\varepsilon_{31}, \varepsilon_{03} = -\frac{1}{2}(\varepsilon_{33} + \varepsilon_{00})$ ³³. One finds

$$K_1 = -\frac{1}{2} \gamma_{01} \left\{ k^0 [(-ip_0 - ip_3)(x^3 - z^3) - 1] - k^3 [(ip_3 + ip_0)(x^0 - z^0) - 1] \right\} \quad (26)$$

$$K_2 = -\frac{1}{2} \gamma_{02} \left\{ k^0 [(-ip_0 - ip_3)(x^3 - z^3) - 1] - k^3 [(ip_3 + ip_0)(x^0 - z^0) - 1] \right\}.$$

It then follows from $K_1 \pm iK_2$ that the helicity is ± 1 . On the contrary K_0, K_3 depend only on ε_{00} and ε_{33} with helicity zero.

Pursuing the "electromagnetic" analogy, one can also derive the equations of motion of a classical particle. On account of (13) and writing $\eta^\alpha \equiv x^\alpha - z^\alpha$, one finds

$$\frac{d^2 \eta^\mu}{ds^2} = F^\mu{}_\lambda u^\lambda \equiv R^\mu{}_{\beta\lambda\alpha} \eta^\alpha u^\beta u^\lambda, \quad (27)$$

which is the equation of geodesic deviation. Similarly, the equation that relates the torque to the change in angular momentum is

$$\eta_{\mu\alpha\beta\rho} \frac{du^\alpha}{ds} \eta^\beta = \eta_{\mu\alpha\beta\rho} F^\alpha{}_\tau u^\tau \eta^\beta \equiv \eta_{\mu\alpha\beta\rho} \eta^\beta \eta^\omega R^\alpha{}_{\sigma\omega\tau} u^\sigma u^\tau. \quad (28)$$

5. Conclusions

The main results are summarized below.

Any linear theory of gravity whose interaction with quantum particles is described by a covariant wave equation leads to a vector theory in which angular momentum and momentum are conserved locally. The ten components of $\gamma_{\mu\nu}(x)$ regroup to form a two-point vector $K_\mu(z, x)$ that satisfies Maxwell-type equations identically. *The result is independent of the choice of any particular set of field equations for the metric tensor* and therefore applies to a variety of theories of gravity, in particular to general relativity and to Caianiello's theory of maximal acceleration. In general, knowledge of $K_\mu(z, x)$ still requires knowledge of all components of $\gamma_{\mu\nu}(x)$. The construction of K_μ can be extended to all orders of approximation in the metric deviation.

The field K_λ renders the local conservation of angular momentum and momentum possible. This is borne out of the integral laws (18) and (21) that state that the flux of $J_{\alpha\beta}$ across a closed V_2 is conserved. This is a general feature of gauge theories of gravity that follows³ from local Poincaré symmetry. If, in particular, the particle wave function is nodal, then the flux of K_λ out of an open V_2 is quantized.

The matter considered is a spinless particle. The introduction of spin^{6,7,8,9} necessitates that $J^{\alpha\beta}$ refer to the total angular momentum in agreement with³.

All results are invariant under the transformations (22) and (25) and cannot therefore be an artifact of the choice of coordinates. In the case $\gamma_{\mu\nu}(x)$ represents a gravitational wave, the helicity of $K_\mu(z, x)$ is that expected of a (not necessarily free) vector field.

The "electromagnetic" analogy can also be pursued at the level of the particle equations of motion. It yields in this instance the equation of geodesic deviation and a generalized torque-change of angular momentum relation.

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